Convex relaxations of mixed derivatives of multivariate entire functions/polynomials

Leonid Gurvits *

Abstract

Many "hard" combinatorial and geometric quantities (such as the number of perfect matchings in bipartite graphs (the **permanent**), number of perfect matchings in general graphs (the **hafnian**) , the number of matching of the fixed size, the number of Hamiltonian cycles, the number of exact 3-coverings, the number of common bases in the intersection of unimodular geometric matroid and the matroid of transversals (the **Mixed Discriminant**), the **Mixed Volume** (responsible for the number of isolated solutions of systems of polynomial equations) etc.) can be expressed as the coefficient $a_{1,...,1} = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(0,0,...,0)$ of the monomial $x_1...x_n$ in some effectively computable "generating" homogeneous polynomial $p \in Hom_+(n,n)$ of degree n with nonnegative rational coefficients. In other words the counting problems are particular cases of partial (for we are after only one coefficient) multivariate interpolations; and the corresponding decision problems(existence of perfect matchings , existence of Hamiltonian cycles etc.) are equivalent to checking if "$\frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(0,0,...,0) = 0$?".
Define the next quantity, associated with a homogeneous polynomial $p \in Hom_+(n,n)$:

$$Cap(p) =: \inf_{x_i > 0} \frac{p(x_1, \dots, x_n)}{\prod_{1 \le i \le n} x_i}.$$

Note that $\log(Cap(p)) = \inf_{y_1+...+y_n=0} \log(p(e^{y_1},...,e^{y_n}))$ and the functional $\log(p(e^{y_1},...,e^{y_n}))$ is convex. (*I will explain that the monotonicity of the celebrated Baum-Welsh algorithm for* **HMM** *is due to the above convexity.*)
Therefore $\log(Cap(p))$ can be, with some extra care and luck, effectively additively approximated using convex programming tools and an oracle, deterministic or random, evaluating the polynomial p. It directly follows from the nonnegativity of the coefficients of p that

$$\log(Cap(p)) \ge \log(\frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(0,0,...,0)).$$

In order to show that $\log(Cap(p))$ is a reasonable relaxation for $\log(\frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(0,0,...,0))$ it is desirable to prove the (optimal) reverse bound:

$$\frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(0,0,...,0) \ge Const(n,p) Cap(p), Const(n,p) > 0.$$

Such lower bound is impossible in general, but I will describe a rather broad class of polynomials(and even of entire functions) for which it is possible.

The combinatorial ([1],[2]), [4], [3]), algorithmic ([5]) and "structural" [6] applications will be described. Most importantly, a fairly complete proof will be presented.

*gurvits@lanl.gov. Los Alamos National Laboratory, Los Alamos, NM.

References

[1] A. Schrijver, Counting 1-factors in regular bipartite graphs, Journal of Combinatorial Theory, Series B 72 (1998) 122–135.

[2] M. Laurent and A. Schrijver, On Leonid Gurvits' proof for permanents, 2009, http://homepages.cwi.nl/ lex/files/perma5.pdf, to appear in *American Mathematical Monthly.*

[3] S. Friedland and L. Gurvits, Lower Bounds for Partial Matchings in Regular Bipartite Graphs and Applications to the Monomer-Dimer Entropy, *Combinatorics, Probability and Computing*, 2008.

[4] L. Gurvits, Van der Waerden/Schrijver-Valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials: one theorem for all, Electronic Journal of Combinatorics 15 (2008).

[5] L. Gurvits, A polynomial-time algorithm to approximate the mixed volume within a simply exponential factor. *Discrete Comput. Geom.* 41 (2009), no. 4, 533–555. On multivariate Newton-like inequalities

[6] L. Gurvits, On multivariate Newton-like inequalities, http://arxiv.org/abs/0812.3687.